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Explicit modular invariant two-loop superstring amplitude relevant for R^4

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Abstract: In this note we derive an explicit modular invariant formula for the two loop 4-point amplitude in superstring theory in terms of a multiple integral (7 complex integration variables) over the complex plane which is shown to be convergent. We consider in particular the case of the leading term for vanishing momenta of the four graviton amplitude, which would correspond to the two-loop correction of the R^4 term in the effective Action. The resulting expression is not positive definite and could be zero, although we cannot see that it vanishes.

Keywords: Superstrings and Heterotic Strings, M-Theory.

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Contents

1. Introduction and summary

Inspired by a recent argument [1, 2] about the R^4 term in superstring theory, it is necessary to reconsider the finiteness of the superstring two-loop 4-graviton amplitude [3], especially for the case of vanishing momenta which was not discussed in detail previously in [3]. The arguments given in [3] were not complete. We can extend these arguments and make them precise [4]. However we will take a different route here.

First we derive an explicit modular invariant formula for the two-loop 4-point amplitude. We remind that the genus-2 Riemann surface, which is the appropriate world sheet for two loops, can be described in full generality by means of the hyperelliptic formalism. This is based on a representation of the surface as two sheet covering of the complex plane described by the equation:

$$y^2(z) = \prod_{i=1}^{6} (z - a_i)\,, \tag{1.1}$$

The complex numbers a_i, $(i = 1, \ldots, 6)$ are the six branch points, by going around them one passes from one sheet to the other. Three of them represent the moduli of the genus 2 Riemann surface over which the integration is performed, while the other three can be arbitrarily fixed. In fact, the two-loop formula, which we obtained by implementing in the hyperelliptic formalism the general algorithm [5] for a multi-genus worldsheet, is expressed as an integral over the Riemann surface moduli, i.e. three branch points, and over the four (complex) position of the vertex operators. The whole integrand, including the measure, is $\mathrm{SL}(2,\mathbb{C})$ invariant and thus three points (i.e. the remaining branch points) are arbitrarily fixed. Modular invariance in this language amounts to invariance under permutations of a_i, $(i = 1,\ldots,6)$, and it is not explicitely apparent in the expression for the amplitude given in [3]. One of the main goal of this paper is to get it manifest.

Our strategy is the following. In our previous calculation we fixed 3 branch points by using the $\mathrm{SL}(2,\mathbb{C})$ invariance of the hyperelliptic representation of genus 2 Riemann surface. Here instead (see sect. 2) we choose to fix 3 out of the 4 vertex operator points and integrate all the 6 branch points. We do it by making a suitable $\mathrm{SL}(2,\mathbb{C})$ change of variables on our previous formula. Amazingly an explicit modular invariant formula is obtained for the 4-point amplitude. This formula is like the Koba-Nielsen formula for the tree amplitude in string theory, in that it is in fact represented as an integral over the whole complex plane of one of the four vertex positions (the other three being fixed). Furthermore there is also the integration over the whole complex plane of the six branch points which appear symmetrically in the integrand.

With this new formula in hand we study its finiteness property by making a convenient choice of the two arbitrary parameters from supercurrent insertion points (some basic facts about this ingredient of the construction of the amplitude are recalled in sect. 2.1). We get thus the following formula for the zero momentum limit of the four graviton amplitude in Type II (A or B) Superstring theory ($z_{1,2,3} = z^0_{1,2,3}$ arbitrarily fixed):

$$AII_0 = c_{II}\, K\, V \int \frac{\prod_{i=1}^6 d^2a_i}{T^5 \prod_{i<j}^6 |a_{ij}|^2}\, \frac{d^2z_4}{|y(z_4)|^2}\, \frac{(z^0_1 - z_4)(\bar{z}^0_2 - \bar{z}_4)}{|y(z^0_1)y(z^0_2)y(z^0_3)|^2} \times \tag{1.2}$$
$$\times \left(\frac{1}{2}\sum_{i=1}^{6}\frac{1}{z^0_1 - a_i} - \sum_{l\neq 1}^{4}\frac{1}{z^0_1 - z_l}\right)\left(\frac{1}{2}\sum_{i=1}^{6}\frac{1}{\bar{z}^0_2 - \bar{a}_i} - \sum_{l\neq 2}^{4}\frac{1}{\bar{z}^0_2 - \bar{z}_l}\right),$$

where T (the determinant of the "period matrix" of the surface) is a function of $a_i, \bar{a}_i$ given in eq. (2.2) below, K is the standard 4-graviton kinematical factor [6, 3] (see appendix B for details and its relation with the R^4 term) and V is a constant factor:

$$V = -\frac{1}{16}|z^0_1 - z^0_2|^4 (z^0_1 - z^0_3)^2 (\bar{z}^0_1 - \bar{z}^0_3)(z^0_2 - z^0_3)(\bar{z}^0_2 - \bar{z}^0_3)^2\,. \tag{1.3}$$

The constant c_{II} in front of the amplitude in eq. (1.2) can be determined by the unitarity relations which for generic momenta relate the two-loop expression with the one-loop and the tree level amplitudes, and thus it is nonzero.

One can check that the integrand of the above formula is $\mathrm{SL}(2,\mathbb{C})$ invariant (including the measure), i.e. invariant under a simultaneous transformation of all the a_i's and z_l's of the kind $w \to (aw + b)/(cw + d)$ with $ad - bc = 1$. Every factor transforms nontrivially and it is rather amazing to see how the whole formula is calibrated. It is also manifestly invariant under permutations of the six branch points $a_{1,\ldots,6}$, i.e. it is manifestly modular invariant. In sect. 3 we make a thorough study of all the possible boundaries of the moduli space and prove that the amplitude is finite, i.e. it is expressed as a convergent 7-(complex)-dimensional integral.

It is not clear whether the result of the integration is zero as it should be in order to agree with the arguments given in refs. [1, 2] about the absence of n-loop (with n larger than 1) contributions to the R^4 term. The above expression for the integrand is certainly not positive definite (we will see in sect. 2.2 that attemping to get a positive definite integrand by adding some total derivative, as it was uncorrectly argued in [3], is not possible due to boundary terms). Still it does not appear to be zero in an obvious way. This can be contrasted with the fact that using the same formalism we have verified other nonrenormalization theorems [7], namely the vanishing of the amplitude with less than 4 vertices. In those cases the integrand itself was found to be zero.

2. A modular invariant form of the amplitude

2.1 Heterotic string amplitude

To simplify the presentation we will first discuss the amplitude for heterotic string. The two loop 4-point amplitude in heterotic string theory obtained in [3, 8] is

$$AH_0 = c_H\, K_H \int \frac{d^2a_1 d^2a_2 d^2a_3\, |a_{45}\, a_{46}\, a_{56}|^2}{T^5 \prod_{i<j}^6 |a_{ij}|^2} \prod_{l=1}^{4} \frac{d^2 z_l (r - z_l)}{y(z_l)} I(r) \bar{F}(\bar{a}, \bar{z})\,, \tag{2.1}$$

where K_H is a kinematic factor depending on the left sector particle contents and $\bar{F}(\bar{a}, \bar{z})$ is a function denoting the contribution from left sector of the heterotic string theory. The complex numbers a_i, $(i = 1, \ldots, 6)$ and the function $y(z)$ which describe the Riemann surface have been introduced in the first section, see eq. (1.1). Three of the six branch points, say $a_{1,2,3}$, represent the moduli of the genus 2 Riemann surface over which the integration is performed, while $a_{4,5,6}$ are arbitrarily fixed. Modular invariance in this language amounts to invariance under permutations of a_i, $(i = 1, \ldots, 6)$, and it is not explicitely apparent in eq. (2.1). One of the main goal of this paper is to get it manifest. Let us remind that eq. (2.1) is obtained after summing over the even spin structures with signs uniquely fixed by the requirement of modular invariance [9]. There is no contribution from the odd spin structure. In fact, the four vertices and the two supercurrent insertion (which we recall below) could just provide the required ten fermionic zero modes, resulting in a contribution proportional to the completely antisymmetric ten dimensional tensor. Since it will be contracted with the four momenta, which are linearly dependent, the result will be identically zero.

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We also recall that $\bar{F}(\bar{a}, \bar{z})$ is symmetric under permutations of $\bar{a}_i$, $(i = 1, \ldots, 6)$. The other functions appearing in (2.1) are

$$T(a_i, \bar{a}_i) = \int \frac{d^2 z_1 d^2 z_2 |z_1 - z_2|^2}{|y(z_1) y(z_2)|^2} , \tag{2.2}$$

which is proportional to the determinant of the period matrix (see ref. [8]), and

$$\begin{aligned} I(r) = & -\frac{1}{2} \sum_{i<j}^{6} \frac{1}{r - a_i} \frac{1}{r - a_j} - \frac{1}{4} \sum_{i<j}^{3} \frac{1}{r - a_i} \frac{1}{r - a_j} + \\ & + \frac{1}{8} \left(\sum_{i=1}^{6} \frac{1}{r - a_i} - 2 \sum_{i=1}^{3} \frac{1}{r - a_i} \right) \sum_{l=1}^{4} \frac{1}{r - z_l} + \\ & + \frac{1}{4} \sum_{i=1}^{6} \frac{1}{r - a_i} \sum_{i=1}^{3} \frac{1}{r - a_i} - \frac{5}{4} \sum_{i=1}^{6} \frac{1}{r - a_i} \frac{\partial}{\partial a_i} \ln T , \end{aligned} \tag{2.3}$$

As we proved in [3, 8] the amplitude AH_0 is independent of the parameter r. This parameter r represents the arbitrary point of insertion of the two supercurrents (also called picture-changing operators) as appropriate for genus 2 where there are two zero modes of the β superghost. We have chosen to insert those two supercurrents at the same point r in the upper and lower sheet of the double covering of the complex plane described by eq. (1.1). The independence of the result from r is in agreement with a general derivation in ref. [5] where it is shown in general that a displacement of r amounts to an irrelevant total derivative in the moduli space.

In fact, we note that $I(r)$ is a rational function of r. By a standard argument from complex analysis or by explicit computation we have

$$I(r) \prod_{l=1}^{4} (r - z_l) = \sum_{i=1}^{6} I_i(r) \prod_{l=1}^{4} (a_i - z_l) + I_\infty , \tag{2.4}$$

where I_∞ is independent of r (see eq. (A.2) in appendix A) and $I_i(r)$ contains the pole singularities for $r \to a_i$.

As we proved in [3], all these singular terms give total derivatives when we insert eq. (2.4) into the 4-point amplitude (2.1). For instance the formula relevant for $I_i(r)$, $(i = 1, 2, 3)$ is (see the appendix A for further details):

$$I_i(r) \prod_{l=1}^{4} \frac{(a_i - z_l)}{y(z_l)} \frac{1}{T^5 \prod_{k<l}^{6} a_{kl}} = \frac{1}{4} \frac{\partial}{\partial a_i} \left\{ \frac{1}{r - a_i} \frac{1}{T^5 \prod_{k<l}^{6} a_{kl}} \prod_{l=1}^{4} \frac{(a_i - z_l)}{y(z_l)} \right\} , \tag{2.5}$$

Similar formulas hold also for $I_i(r)$, $(i = 4, 5, 6)$ (eq. (A.7) in appendix A). These total derivative terms give zero contribution when integrated over the moduli as in eq. (2.1), since we proved in [3, 8] that there are no possible boundary contributions. Therefore AH_0 is independent from r.

Next, we note that the amplitude (2.1) is $\mathrm{SL}(2,\mathbb{C})$ invariant, i.e. a simultaneous $\mathrm{SL}(2,\mathbb{C})$ transformation on all the integration variables $a_{1,2,3}$, $z_{1,2,3,4}$ and the fixed point $a_{4,5,6}$ and r leaves AH_0 invariant, i.e.

$$AH_0 = c_H\, K_H \int \frac{d^2S(a_1)d^2S(a_2)d^2S(a_3)\ \prod_{i<j=4}^{6}|S(a_i)-S(a_j)|^2}{T^5\prod_{i<j}^{6}|S(a_i)-S(a_j)|^2}\times$$
$$\times\prod_{l=1}^{4}\frac{d^2S(z_l)(S(r)-S(z_l))}{y(S(z_l))}I(S(r))\bar{F}(\bar{S}(\bar{a}),\bar{S}(\bar{z}))\,, \qquad (2.6)$$

where S denotes an $\mathrm{SL}(2,\mathbb{C})$ transformation. Here the functions T, $y(S(z))$ and $I(S(r))$ have in their definitions with a_i changed to $S(a_i)$. $S(x)$ depends on three parameters $q_{1,2,3}$ and thus we set

$$S(x)\equiv S_q(x)=\frac{(q_1-q_2)(x-q_3)}{(q_1-q_3)(x-q_2)}\,. \qquad (2.7)$$

Now we insert the following identity

$$\int\prod_{i=1}^{3}d^2q_i\ \prod_{i=1}^{3}\delta^2(S_q(z_i)-Z_i^0)\ \frac{\prod_{i<j=1}^{3}|Z_i^0-Z_j^0|^2}{\prod_{i<j=1}^{3}|q_i-q_j|^2}=1\,, \qquad (2.8)$$

into eq. (2.6) and we get

$$AH_0 = c_H\, K_H \int \frac{d^2S(a_1)d^2S(a_2)d^2S(a_3)\ \prod_{i<j=4}^{6}|S(a_i)-S(a_j)|^2}{T^5\prod_{i<j}^{6}|S(a_i)-S(a_j)|^2}\times$$
$$\times\prod_{l=1}^{4}\frac{d^2S(z_l)(S(r)-S(z_l))}{y(S(z_l))}\prod_{i=1}^{3}d^2q_i\ \prod_{i=1}^{3}\delta^2(S_q(z_i)-Z_i^0)\times$$
$$\times\frac{\prod_{i<j=1}^{3}|Z_i^0-Z_j^0|^2}{\prod_{i<j=1}^{3}|q_i-q_j|^2}I(S(r))\bar{F}(\bar{S}(\bar{a}),\bar{S}(\bar{z}))\,. \qquad (2.9)$$

Now taking $S=S_q$ and noticing the following relation

$$\prod_{i=1}^{3}dq_i\,\frac{\prod_{i<j=4}^{6}(S_q(a_i)-S_q(a_j))}{\prod_{i<j=1}^{3}(q_i-q_j)}=\prod_{i=4}^{6}dS_q(a_i), \qquad (2.10)$$

we get

$$AH_0 = c_H\, K_H \int \frac{\prod_{i=1}^{6}d^2A_i}{T^5\prod_{i<j}^{6}|A_i-A_j|^2}\prod_{l=1}^{4}\frac{d^2Z_l(S_A(r)-Z_l)}{Y(Z_l)}\times$$
$$\times\prod_{i=1}^{3}\delta^2(Z_i-Z_i^0)\prod_{i<j=1}^{3}|Z_i^0-Z_j^0|^2 I(S_A(r))\bar{F}(\bar{A},\bar{z}))\,, \quad (2.11)$$

where we have renamed $S_q(a_i)$ to be A_i, $S_q(z_l)$ to be Z_l and $Y^2(Z)=\prod_{i=1}^{6}(Z-A_i)$. Also we changed the name of $\mathrm{SL}(2,\mathbb{C})$ transformation of S_q to be S_A, since $q_{1,2,3}$ are fixed in terms of $A_{4,5,6}$. Explicitly we have

$$S_q(r)=S_A(r)=\frac{A_6(A_4-A_5)r-A_4(A_6-A_5)}{(A_4-A_5)r-(A_6-A_5)}\,. \qquad (2.12)$$

From eq. (2.11) we see that because of the 3 δ functions the integration variables $Z_{1,2,3}$ are fixed and all the $A_{1,\dots,6}$ are integrated. Nevertheless this formula is not explicitly modular invariant, i.e. it's not invariant under any interchange of A_i and A_j. In what follows we will put it into a modular invariant form.

Setting

$$I_M(x) = \frac{1}{4}\sum_{i=1}^{6}\frac{1}{(x-A_i)^2} - \frac{1}{4}\sum_{i<j}^{6}\frac{1}{x-A_i}\frac{1}{x-A_j} - \frac{1}{8}\sum_{i=1}^{6}\frac{1}{x-A_i}\sum_{l=1}^{4}\frac{1}{x-Z_l} + \\ +\frac{1}{4}\sum_{k<l=1}^{4}\frac{1}{x-Z_k}\frac{1}{x-Z_l} - \frac{5}{4}\sum_{i=1}^{6}\frac{1}{x-A_i}\frac{\partial}{\partial A_i}\ln T\,, \tag{2.13}$$

we have the following identity

$$I(S_A(r))\prod_{k=1}^{4}\frac{S_A(r)-Z_k}{Y(Z_k)}\frac{1}{T^5\prod_{i<j=1}^{6}A_{ij}} = I_M(x)\prod_{k=1}^{4}\frac{x-Z_k}{Y(Z_k)}\frac{1}{T^5\prod_{i<j=1}^{6}A_{ij}} - \\ -\frac{1}{4}\sum_{i=1}^{6}\frac{\partial}{\partial A_i}\left[\left(\frac{1}{A_i-S_A(r)}-\frac{1}{A_i-x}\right)\prod_{l=1}^{4}\frac{A_i-Z_l}{Y(Z_l)}\frac{1}{T^5\prod_{j<k=1}^{6}A_{jk}}\right]. \tag{2.14}$$

By using this identity in eq. (2.11) and by dropping all the total derivatives terms, we finally get

$$AH_0 = c_H\,K_H\int\frac{\prod_{i=1}^{6}d^2A_i}{T^5\prod_{i<j}^{6}|A_i-A_j|^2}\prod_{l=1}^{4}\frac{d^2Z_l(x-Z_l)}{Y(Z_l)}\times \\ \times\prod_{i=1}^{3}\delta^2(Z_i-Z_i^0)\prod_{i<j=1}^{3}|Z_i^0-Z_j^0|^2\,I_M(x)\bar{F}(\bar{A},\bar{z}))\,. \tag{2.15}$$

Notice that $I_M(x)$ is a modular invariant function, i.e. invariant under any permutation of the branch points A_i, $(i=1,\dots,6)$. So we obtain a modular invariant formula for the two loop 4-particle amplitude in heterotic string theory. It is also independent of the parameter x as one can easily check that changing x amounts to irrelevant total derivatives, as we have done in appendix A.

2.2 Type II superstring amplitude

For type II superstring theory, we can use the same method to get a modular invariant formula. Let us start with the two loop 4-point amplitude (with vanishing momenta) in type II superstring theory [3, 8]:

$$AII_0 = c_{II}\,K\int\frac{d^2a_1d^2a_2d^2a_3\,|a_{45}\,a_{46}\,a_{56}|^2}{T^5\prod_{i<j}^{6}|a_{ij}|^2}\prod_{l=1}^{4}\frac{d^2z_l(r-z_l)(\bar{s}-\bar{z}_l)}{|y(z_l)|^2}\times \\ \times\left\{I(r)\bar{I}(\bar{s})+\frac{5}{4}\left(\frac{\pi}{T\,y(r)\bar{y}(\bar{s})}\int\frac{d^2w(r-w)(\bar{s}-\bar{w})}{|y(w)|^2}\right)^2\right\}, \tag{2.16}$$

where K is the standard 4-graviton kinematical factor [6, 3]. The complete amplitude and K factor is given in appendix B. Now there are two arbitrary parameters from supercurrent-insertion points, r from the right sector and $\bar{s}$ from the left sector. Again, the resulting amplitude AII_0 is independent of them.

Now we perform the same trick of performing a $\mathrm{SL}(2,\mathbb{C})$ transformation on (2.16) as in the heterotic string case and obtain the following:

$$\begin{aligned} AII_0 = c_{II}\, K \int & \frac{\prod_{i=1}^{6} d^2 A_i}{T^5 \prod_{i<j}^{6} |A_{ij}|^2} \prod_{l=1}^{4} \frac{d^2 Z_l (S_A(r) - Z_l)(\bar{S}_A(\bar{s}) - \bar{Z}_l)}{|Y(Z_l)|^2} \times \\ & \times \prod_{i=1}^{3} \delta^2(Z_i - Z_i^0) \prod_{i<j=1}^{3} |Z_i^0 - Z_j^0|^2 \Big\{ I(S_A(r)) \bar{I}(\bar{S}_A(\bar{s}) + \\ & + \frac{5}{4} \left(\frac{\pi}{T\, Y(S_A(r)) \bar{Y}(\bar{S}_A(\bar{s}))} \int \frac{d^2 v (S_A(r) - v)(\bar{S}_A(\bar{s})) - \bar{v})}{|Y(v)|^2} \right)^2 \Big\}. \end{aligned} \tag{2.17}$$

To proceed further we need some formulas reported in the appendix B.

By using eq. (B.11) of the appendix B in (2.17), it is now easy to derive an explicit modular invariant formula for type II superstring by dropping all total derivative terms. We get

$$\begin{aligned} AII_0 = c_{II}\, K \int & \frac{\prod_{i=1}^{6} d^2 A_i}{T^5 \prod_{i<j}^{6} |A_{ij}|^2} \prod_{l=1}^{4} \frac{d^2 Z_l (x - Z_l)(\bar{w} - \bar{Z}_l)}{|Y(Z_l)|^2} \prod_{i=1}^{3} \delta^2(Z_i - Z_i^0) \times \\ \times \prod_{i<j=1}^{3} |Z_i^0 - Z_j^0|^2 & \left\{ I_M(x) \bar{I}_M(\bar{w}) + \frac{5}{4} \left(\frac{\pi}{TY(x)\bar{Y}(\bar{w})} \int \frac{d^2 v (x - v)(\bar{w} - \bar{v})}{|Y(v)|^2} \right)^2 \right\}. \end{aligned} \tag{2.18}$$

where $I_M(x)$ is the same as in eq. (2.13). Again, x and $\bar{w}$ are arbitrary and the result does not depend on them.

We note that we can't set $x = w$ in eq. (2.18) because with $x = w$ in eq. (2.18) AII_0 is divergent. The important point to note here is the following: the total derivative terms (see eq. (B.11)) which would be dropped are actually in this case non-vanishing and divergent. This can be clearly demonstrated by the following computation ($A_i - x = r\, e^{i\theta}$):

$$\begin{aligned} \int_{\epsilon<|A_i - x|<\Lambda} & d^2 A_i \frac{\partial}{\partial A_i} \left[\frac{1}{A_i - x} f(A, \bar{A}) \bar{I}_M(\bar{w}) \right] = \\ & = \int_\epsilon^\Lambda r\, d\, r \int_0^{2\pi} d\theta\, e^{-i\theta} \left(\frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \theta} \right) \left[\frac{1}{r\, e^{i\theta}} f(A, \bar{A}) \bar{I}_M(\bar{w}) \right] \\ & = \int_\epsilon^\Lambda d\, r \int_0^{2\pi} d\theta \left(e^{-2i\theta} \frac{\partial}{\partial r} f(A, \bar{A}) \bar{I}_M(\bar{w}) + \frac{1}{r} \frac{\partial}{\partial \theta} \left[e^{-2i\theta} f(A, \bar{A}) \bar{I}_M(\bar{w}) \right] \right). \end{aligned} \tag{2.19}$$

In the above equation the last term is a total derivative in θ and can be dropped. On the other hand the first term is a total derivative in r and its integration over r gives two terms evaluated at the points $r = \epsilon$ and $r = \Lambda$:

$$\int_0^{2\pi} d\theta \left(e^{-2i\theta} f(A, \bar{A}) \bar{I}_M(\bar{w})|_{r=\Lambda} - e^{-2i\theta} f(A, \bar{A}) \bar{I}_M(\bar{w})|_{r=\epsilon} \right). \tag{2.20}$$

Here we only need to consider the second term because the first term is just an artifact of cutoff. When $w \neq x$, the second term is regular for $r = \epsilon \to 0$ and so there is no boundary term. When $w = x$, the term $1/(\bar{A}_i - \bar{w})^2$ from $\bar{I}_M(\bar{w})$ gives a contribution which is $2\pi/\epsilon^2$ after integration over θ which is divergent for $\epsilon \to 0$ while the other terms give non-vanishing but finite contributions.

3. The finiteness of the amplitude

Now we study the finiteness property of the amplitude AII_0 in (2.18). First we make the following choice of the two arbitrary parameters x and w (which have their origin as the insertion point of the supercurrents): $x = Z_1^0$ and $w = Z_2^0$. With this choice the amplitude AII_0 simplifies greatly:

$$\begin{aligned} AII_0 = c_{II}\, K\, V \int \frac{\prod_{i=1}^6 d^2 A_i}{T^5 \prod_{i<j}^6 |A_{ij}|^2} \frac{d^2 Z_4}{|Y(Z_4)|^2} \frac{(Z_1^0 - Z_4)(\bar{Z}_2^0 - \bar{Z}_4)}{|Y(Z_1^0) Y(Z_2^0) Y(Z_3^0)|^2} \times \\ \times \left(\frac{1}{2} \sum_{i=1}^{6} \frac{1}{Z_1^0 - A_i} - \sum_{l \neq 1}^{4} \frac{1}{Z_1^0 - Z_l} \right) \left(\frac{1}{2} \sum_{i=1}^{6} \frac{1}{\bar{Z}_2^0 - \bar{A}_i} - \sum_{l \neq 2}^{4} \frac{1}{\bar{Z}_2^0 - \bar{Z}_l} \right), \end{aligned} \tag{3.1}$$

where V is a constant factor:

$$V = -\frac{1}{16} |Z_1^0 - Z_2^0|^4 (Z_1^0 - Z_3^0)^2 (\bar{Z}_1^0 - \bar{Z}_3^0)(Z_2^0 - Z_3^0)(\bar{Z}_2^0 - \bar{Z}_3^0)^2. \tag{3.2}$$

To prepare the discussion of finiteness of the above amplitude we first note that due to the angular integration, the factor $(Z_1^0 - Z_4)(\bar{Z}_2^0 - \bar{Z}_4)$ times the second line of eq. (3.2) can be omitted. Although this factor may be singular for $A_i \to Z_{1,2}^0$ (it is not singular for $Z_4 \to Z_{1,2}^0$), the singular terms only appear either in the holomorphic part or the anti-holomorphic part. The integration over the angular variable then gives a contribution which is always not more singular than the contribution without this factor. So for the purpose of discussing finiteness we can just consider the finiteness of the truncated expression:

$$AII_T = \int \frac{\prod_{i=1}^6 d^2 A_i}{T^5 \prod_{i<j}^6 |A_{ij}|^2} \frac{d^2 Z_4}{|Y(Z_4)|^2} \frac{1}{|Y(Z_1^0) Y(Z_2^0) Y(Z_3^0)|^2}. \tag{3.3}$$

To discuss the finiteness property of the amplitude AII_T in (3.3) we keep the fixed points Z_i^0 at generic points and classify all the boundaries into various cases.

In the first case $a)$ all the branch points are kept away from the fixed points Z_i^0 and we use the following parametrization:

$$\begin{aligned} A_1 &= A_1\,, \\ A_2 &= A_1 + u\,, \\ A_i &= A_1 + u\, v_{i-2}\,, \qquad i = 3, \ldots, n \leq 6\,, \end{aligned} \tag{3.4}$$

and consider the limit $u \to 0$ and v_i's finite. The integration over all A_i ($i = 1, \ldots, 6$) changes to an integration over u, A_1, v_i and the remaining A_j's.

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In the second case b) a subset of the branch points approach one of the points of Z_i^0 and we use the following parametrization:

$$
\begin{aligned}
A_1 &= Z_i^0 + u\,, \\
A_j &= Z_i^0 + u\, v_{j-1}\,, \qquad j = 2, \ldots, n \leq 6\,,
\end{aligned} \tag{3.5}
$$

and also consider the limit $u \to 0$ and v_i's finite. The integration over all A_i ($i = 1, \ldots, 6$) changes to an integration over u, v_i and the rest A_j's.

The third case is a combination of the first two cases and its finiteness is proved if we can prove the finiteness in the first two cases. In each cases the integration over Z_4 must also be considered to obtain the most singular behaviour.

It is straightforward to compute the behaviour of the various factors under various degeneration limits. In particular the function T under various degeneration limit $u \to 0$ behaves as follows:

$$
T \sim \begin{cases} \ln|u|, & A_2 \to A_1\,, \\ \dfrac{1}{|u|}, & A_{2,3} \to A_1\,, \\ \dfrac{\ln|u|}{|u|^2}, & A_{2,3,4} \to A_1\,, \\ \dfrac{1}{|u|^4}, & A_{2,\ldots,5} \to A_1\,, \\ \dfrac{1}{|u|^6}, & A_{2,\ldots,6} \to A_1\,. \end{cases} \tag{3.6}
$$

In the first case the factor $1/(|Y(Z_1^0)Y(Z_2^0)Y(Z_3^0)|^2)$ is non-singular. The behaviour of the other factors is given in the following table:

	$\dfrac{1}{T^5}$	$\prod_{i=1}^{6} d^2A_i$	$\dfrac{1}{\prod_{i<j}^{6} \lvert A_{ij}\rvert^2}$	$\int \dfrac{d^2Z_4}{\lvert Y(Z_4)\rvert^2}$	All
$A_2 \to A_1$	$\dfrac{1}{(\ln\lvert u\rvert)^5}$	d^2u	$\dfrac{1}{\lvert u\rvert^2}$	$\ln\lvert u\rvert$	$\dfrac{d^2u}{\lvert u\rvert^2}\,\dfrac{1}{(\ln\lvert u\rvert)^4}$
$A_{2,3} \to A_1$	$\lvert u\rvert^5$	$\lvert u\rvert^2 d^2u$	$\dfrac{1}{\lvert u\rvert^6}$	$\dfrac{1}{\lvert u\rvert}$	d^2u
$A_{2,3,4} \to A_1$	$\dfrac{\lvert u\rvert^{10}}{(\ln\lvert u\rvert)^5}$	$\lvert u\rvert^4 d^2u$	$\dfrac{1}{\lvert u\rvert^{12}}$	$\dfrac{1}{\lvert u\rvert^2}$	$d^2u\,\dfrac{1}{(\ln\lvert u\rvert)^5}$
$A_{2,\ldots,5} \to A_1$	$\lvert u\rvert^{20}$	$\lvert u\rvert^6 d^2u$	$\dfrac{1}{\lvert u\rvert^{20}}$	$\dfrac{1}{\lvert u\rvert^3}$	$\lvert u\rvert^3 d^2u$
$A_{2,\ldots,6} \to A_1$	$\lvert u\rvert^{30}$	$\lvert u\rvert^8 d^2u$	$\dfrac{1}{\lvert u\rvert^{30}}$	$\dfrac{1}{\lvert u\rvert^4}$	$\lvert u\rvert^4 d^2u$

In the last column we have put all factors together and one sees quite clearly that the integration over u gives a finite contribution from the corner $u \sim 0$. This proves that the amplitude AII_0 is finite in the first case a) degeneration limit.

In the second case $b)$ only one $Y(Z_i^0)$ in the factor $\frac{1}{|Y(Z_1^0)Y(Z_2^0)Y(Z_3^0)|^2}$ gives singular contribution. This contribution and the behaviour of the other factors is given in the following table:

	$\frac{1}{T^5}$	$\prod_{i=1}^{6} d^2A_i$	$\frac{1}{\prod_{i<j}^{6}\|A_{ij}\|^2}$	$\int \frac{d^2Z_4}{\|Y(Z_4)\|^2}$	$\frac{1}{\|Y(Z_i^0)\|^2}$	All
$A_1 \to Z_i^0$	1	d^2u	1	1	$\frac{1}{\|u\|}$	$\frac{d^2u}{\|u\|}$
$A_{1,2} \to Z_i^0$	$\frac{1}{(\ln\|u\|)^5}$	$\|u\|^2d^2u$	$\frac{1}{\|u\|^2}$	$\ln\|u\|$	$\frac{1}{\|u\|^2}$	$\frac{d^2u/\|u\|^2}{(\ln\|u\|)^4}$
$A_{1,2,3} \to Z_i^0$	$\|u\|^5$	$\|u\|^4d^2u$	$\frac{1}{\|u\|^6}$	$\frac{1}{\|u\|}$	$\frac{1}{\|u\|^3}$	$\frac{d^2u}{\|u\|}$
$A_{1,\dots,4} \to Z_i^0$	$\frac{\|u\|^{10}}{(\ln\|u\|)^5}$	$\|u\|^6d^2u$	$\frac{1}{\|u\|^{12}}$	$\frac{1}{\|u\|^2}$	$\frac{1}{\|u\|^4}$	$\frac{d^2u/\|u\|^2}{(\ln\|u\|)^5}$
$A_{1,\dots,5} \to Z_i^0$	$\|u\|^{20}$	$\|u\|^8d^2u$	$\frac{1}{\|u\|^{20}}$	$\frac{1}{\|u\|^3}$	$\frac{1}{\|u\|^5}$	d^2u
$A_{1,\dots,6} \to Z_i^0$	$\|u\|^{30}$	$\|u\|^{10}d^2u$	$\frac{1}{\|u\|^{30}}$	$\frac{1}{\|u\|^4}$	$\frac{1}{\|u\|^6}$	d^2u

In the last column we have put all factors together and one sees quite clearly that the integration over u gives a finite contribution from the corner $u \sim 0$. This proves that the amplitude AII_0 is also finite in the second case $b)$ degeneration limit. This completes our proof that the modular invariant two-loop 4-particle superstring amplitude is finite.

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A. Independence of the amplitude on the arbitrary parameter

In this appendix we gave some details [3, 8] in proving the independence of the heterotic string amplitude on the parameter r. Let us first recall eq. (2.4) here:

$$I(r)\prod_{l=1}^{4}(r-z_l) = \sum_{i=1}^{6} I_i(r)\prod_{l=1}^{4}(a_i - z_l) + I_\infty, \tag{A.1}$$

where I_∞ is given as follows [3]:

$$\begin{aligned} I_\infty = & -\frac{1}{2}\sum_{i<j}^{6} a_i a_j - \frac{1}{4}\sum_{i<j}^{6} a_i a_j + \frac{1}{8}\left(\sum_{i=1}^{6} a_i - 2\sum_{i=1}^{3} a_i\right)\sum_{l=1}^{4} z_l + \\ & +\frac{1}{4}\sum_{i=1}^{6} a_i \sum_{j=1}^{3} a_j - \frac{5}{4}\sum_{i=1}^{6}\left(a_i^2 + a_i^3\frac{\partial}{\partial a_i}\ln T\right), \end{aligned} \tag{A.2}$$

and $I_i(r)$ are defined as the singular terms of $I(r)$ times some factors as $r \to a_i$:

$$I(r) \prod_{l=1}^{4}(r - z_l) \to I_i(r) \prod_{l=1}^{4}(a_i - z_l)\,, \qquad \text{for } r \to a_i\,. \tag{A.3}$$

Explicitly we have:

$$\begin{aligned} I_i(r) &= \frac{1}{4}\frac{1}{(r-a_i)^2} - \frac{1}{4}\frac{1}{r-a_i}\sum_{j\neq i}\frac{1}{a_i - a_j} + \\ &\quad + \frac{1}{8}\frac{1}{r-a_i}\sum_{l=1}^{4}\frac{1}{a_i - z_l} - \frac{5}{4}\frac{1}{r-a_i}\frac{\partial}{\partial a_i}\ln T\,, \qquad i = 1,2,3\,, \qquad (A.4) \\ I_4(r) &= \frac{1}{r-a_4}\left(-\frac{1}{2}\left(\frac{1}{a_4-a_5} + \frac{1}{a_4-a_6}\right) - \right. \\ &\qquad \left. -\frac{1}{4}\sum_{i=1}^{3}\frac{1}{a_4-a_i} + \frac{1}{8}\sum_{l=1}^{4}\frac{1}{a_4 - z_l} - \frac{5}{4}\frac{\partial}{\partial a_4}\ln T\right), \quad \text{etc.}\ldots \qquad (A.5) \end{aligned}$$

In deriving eq. (A.1) we have used the property of T under the $\mathrm{SL}(2,\mathbb{C})$ linear fractional transformation. It gives the following identities:

$$\sum_{i=1}^{6} a_i^n \frac{\partial}{\partial a_i}\ln T = \begin{cases} 0\,, & n = 0\,, \\ -3\,, & n = 1\,, \\ -\sum_{i=1}^{6} a_i\,, & n = 3\,. \end{cases} \tag{A.6}$$

Now we prove that all terms containing $I_i(r)$ give total derivative terms. The formula for $I_i(r)$ $(i = 1,2,3)$ is already given in eq. (2.5) and the formula for $I_4(r)$ is

$$\begin{aligned} I_4(r)\prod_{l=1}^{4}\frac{(a_4-z_l)}{y(z_l)}\frac{1}{T^5\prod_{k<l}^{6}a_{kl}} &= \frac{1}{4}\sum_{i=1}^{3}\frac{\partial}{\partial a_i}\left\{\frac{1}{r-a_4}\frac{a_{i5}a_{i6}}{a_{45}a_{46}}\frac{1}{T^5\prod_{k<l}^{6}a_{kl}}\prod_{l=1}^{4}\frac{(a_4-z_l)}{y(z_l)}\right\} + \\ &+ \frac{1}{4}\sum_{l=1}^{4}\frac{\partial}{\partial z_l}\left\{\frac{1}{(r-a_4)}\frac{(a_5-z_l)(a_6-z_l)}{a_{45}a_{46}}\frac{1}{T^5\prod_{k<l}^{6}a_{kl}}\prod_{l=1}^{4}\frac{(a_4-z_l)}{y(z_l)}\right\}. \end{aligned} \tag{A.7}$$

In the process of deriving (A.7) we used (A.6) again to express the derivative of T with respect to a_4 in terms of the derivatives of a_i $(i = 1,2,3)$:

$$\frac{\partial}{\partial a_4}\ln T = \frac{1}{a_{45}a_{46}}\left\{2(a_5+a_6) - \sum_{i=1}^{4}a_i - \sum_{i=1}^{3}a_{i5}a_{i6}\frac{\partial}{\partial a_i}\ln T\right\}. \tag{A.8}$$

B. 4-graviton amplitude and R^4 term

In this appendix we give the complete 4-graviton amplitude, the kinematic factor and collect some formulas used in the discussion of the Type II Superstrig case. The vertex operator for the emission of a graviton with momentum k^μ and polarization tensor $\epsilon^{\mu\nu}$ is

$$\begin{aligned} V(k,\epsilon;z,\bar{z}) = \epsilon^{\mu\nu} : &\Big(\partial_z X_\mu(z,\bar{z}) + ik\cdot\psi(z)\psi_\mu(z)\Big) \times \\ &\times\Big(\partial_{\bar{z}} X_\nu(z,\bar{z}) + ik\cdot\tilde{\psi}(\bar{z})\tilde{\psi}_\nu(\bar{z})\Big)\, e^{i\cdot kX(z,\bar{z})} :\,. \end{aligned} \tag{B.1}$$

The complete two-loop 4-graviton amplitude is

$$
\begin{aligned}
AII(k_i,\epsilon_i) = c_{II}\, K \int \frac{d^2a_1 d^2a_2 d^2a_3\, |a_{45}\, a_{46}\, a_{56}|^2}{T^5 \prod_{i<j}^6 |a_{ij}|^2} \prod_{l=1}^{4} \frac{d^2 z_l (r-z_l)(\bar{s}-\bar{z}_l)}{|y(z_l)|^2} \times \\
\times \Bigg\{ \left(I(r)\bar{I}(\bar{s}) + \frac{5}{4}\left(\frac{\pi}{T\, y(r)\bar{y}(\bar{s})}\int \frac{d^2w(r-w)(\bar{s}-\bar{w})}{|y(w)|^2}\right)^2\right) \langle \prod e^{ik\cdot X}\rangle + \\
+ \frac{1}{16}\left\langle \partial X(r+)\cdot\partial X(r-)\bar{\partial}X(\bar{s}+)\cdot\bar{\partial}X(\bar{s}-)\prod e^{ik\cdot X}\right\rangle - \\
- \frac{1}{16}\left\langle \partial X(r+)\cdot\partial X(r-)\bar{\partial}X(\bar{s}+)\cdot\bar{\partial}X(\bar{s}-)\right\rangle \left\langle \prod e^{ik\cdot X}\right\rangle \Bigg\},
\end{aligned}
\tag{B.2}
$$

where c_{II} is an overall constant which should be fixed by unitarity and K is the standard kinematic factor (for $\epsilon_i^{\mu\nu} = \epsilon_i^{\mu}\tilde{\epsilon}_i^{\nu}$) [6, 3]:

$$
K = K_R \cdot K_L, \tag{B.3}
$$

$$
\begin{aligned}
K_R = & -\frac{1}{4}(st\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4 + su\epsilon_2\cdot\epsilon_3\epsilon_1\cdot\epsilon_4 + tu\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4) + \\
& +\frac{1}{2}\, s\,(\epsilon_1\cdot k_4\epsilon_3\cdot k_2\epsilon_2\cdot\epsilon_4 + \epsilon_2\cdot k_3\epsilon_4\cdot k_1\epsilon_1\cdot\epsilon_3 + \\
& \quad + \epsilon_1\cdot k_3\epsilon_4\cdot k_2\epsilon_2\cdot\epsilon_3 + \epsilon_2\cdot k_4\epsilon_3\cdot k_1\epsilon_1\cdot\epsilon_4) + \\
& +\frac{1}{2}\, t\,(\epsilon_2\cdot k_1\epsilon_4\cdot k_3\epsilon_1\cdot\epsilon_3 + \epsilon_3\cdot k_4\epsilon_1\cdot k_2\epsilon_2\cdot\epsilon_4 + \\
& \quad + \epsilon_2\cdot k_4\epsilon_1\cdot k_3\epsilon_3\cdot\epsilon_4 + \epsilon_3\cdot k_1\epsilon_4\cdot k_2\epsilon_1\cdot\epsilon_2) + \\
& +\frac{1}{2}\, u\,(\epsilon_1\cdot k_2\epsilon_4\cdot k_3\epsilon_2\cdot\epsilon_3 + \epsilon_3\cdot k_4\epsilon_2\cdot k_1\epsilon_1\cdot\epsilon_4 + \\
& \quad + \epsilon_1\cdot k_4\epsilon_2\cdot k_3\epsilon_1\cdot\epsilon_4 + \epsilon_3\cdot k_2\epsilon_4\cdot k_1\epsilon_1\cdot\epsilon_2)\,,
\end{aligned}
\tag{B.4}
$$

$$
K_L = K_R(\epsilon\to\tilde{\epsilon})\,. \tag{B.5}
$$

Here s, t and u are the standard Mandelstam variables for the 4-gravitons. Defining $t_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}$ as follows:

$$
K_R = t_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\epsilon_1^{\mu_1}k_1^{\nu_1}\epsilon_2^{\mu_2}k_2^{\nu_2}\epsilon_3^{\mu_3}k_3^{\nu_3}\epsilon_4^{\mu_4}k_4^{\nu_4}\,, \tag{B.6}
$$

the R^4 term is given as follows:

$$
R^4 = t^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}t^{\rho_1\sigma_1\rho_2\sigma_2\rho_3\sigma_3\rho_4\sigma_4}R_{\mu_1\nu_1\rho_1\sigma_1}R_{\mu_2\nu_2\rho_2\sigma_2}R_{\mu_3\nu_3\rho_3\sigma_3}R_{\mu_4\nu_4\rho_4\sigma_4}\,. \tag{B.7}
$$

To prove the independence of the amplitude AII on r and $\bar{s}$ in eq. (2.18) we need the following identity:

$$
\frac{1}{4}\sum_{j=1}^{6}\frac{1}{\bar{A}_j-\bar{s}}\frac{\partial^2}{\partial\bar{A}_j\partial A_i}\ln T = \frac{1}{\prod_{j\neq i}^{6}(A_i-A_j)}\left(\frac{\pi}{T\,\bar{Y}(\bar{s})}\int\frac{d^2v(A_i-v)(\bar{s}-\bar{v})}{|Y(v)|^2}\right)^2, \tag{B.8}
$$

which was proved in [8]. Notice also the following identity

$$\begin{aligned}\frac{\prod_{l=1}^{6}(r-Z_l)}{Y^2(r)} &= 1+\sum_{i=1}^{6}\frac{1}{r-A_i}\,\frac{\prod_{l=1}^{6}(A_i-Z_l)}{\prod_{j\neq i}^{6}(A_i-A_j)} \\ &= \frac{\prod_{l=1}^{6}(x-Z_l)}{Y^2(x)}+\sum_{i=1}^{6}\left(\frac{1}{r-A_i}-\frac{1}{x-A_i}\right)\frac{\prod_{l=1}^{6}(A_i-Z_l)}{\prod_{j\neq i}^{6}(A_i-A_j)}\,,\end{aligned} \tag{B.9}$$

we have

$$\begin{aligned}&\left(\frac{\pi}{T\,Y(r)\bar{Y}(\bar{s})}\int\frac{d^2v(r-v)(\bar{s}-\bar{v})}{|Y(v)|^2}\right)^2\prod_{l=1}^{4}(r-Z_l)= \\ &=\left(\frac{\pi}{T\,Y(x)\bar{Y}(\bar{s})}\int\frac{d^2v(x-v)(\bar{s}-\bar{v})}{|Y(v)|^2}\right)^2\prod_{l=1}^{4}(x-Z_l)+ \\ &\quad+\sum_{i=1}^{6}\left(\frac{1}{r-A_i}-\frac{1}{x-A_i}\right)\frac{\prod_{l=1}^{4}(A_i-Z_l)}{\prod_{j\neq i}^{6}(A_i-A_j)}\left(\frac{\pi}{T\,\bar{Y}(\bar{s})}\int\frac{d^2v(A_i-v)(\bar{s}-\bar{v})}{|Y(v)|^2}\right)^2 \\ &=\left(\frac{\pi}{T\,Y(x)\bar{Y}(\bar{s})}\int\frac{d^2v(x-v)(\bar{s}-\bar{v})}{|Y(v)|^2}\right)^2\prod_{l=1}^{4}(x-Z_l)+ \\ &\quad+\frac{1}{4}\sum_{i=1}^{6}\prod_{l=1}^{4}(A_i-Z_l)\left(\frac{1}{r-A_i}-\frac{1}{x-A_i}\right)\sum_{j=1}^{6}\frac{1}{\bar{A}_j-\bar{s}}\,\frac{\partial^2}{\partial\bar{A}_j\partial A_i}\ln T,\end{aligned} \tag{B.10}$$

by making use of (B.8).

By using eq. (2.14) for both the left sector and right sector and also the above equation with $r\to S_A(r)$ we have

$$\begin{aligned}&\left\{I(S_A(r))\bar{I}(\bar{S}_A(\bar{s}))+\frac{5}{4}\left(\frac{\pi}{T}\int\frac{d^2v(S_A(r)-v)(\bar{S}_A(\bar{s})-\bar{v})}{Y(S_A(r))\bar{Y}(\bar{S}_A(\bar{s}))|y(v)|^2}\right)^2\right\}\times \\ &\qquad\times\prod_{k=1}^{4}\frac{(S_A(r)-Z_k)(\bar{S}_A(\bar{s})-\bar{Z}_k)}{|Y(Z_k)|^2}\,\frac{1}{T^5\prod_{i<j=1}^{6}|A_{ij}|^2}= \\ &\qquad=\left\{I_M(x)\,\bar{I}_M(\bar{w})+\frac{5}{4}\left(\frac{\pi}{T\,Y(x)\bar{Y}(\bar{w})}\int\frac{d^2v(x-v)(\bar{w}-\bar{v})}{|Y(v)|^2}\right)^2\right\}\times \\ &\qquad\quad\times\prod_{k=1}^{4}\frac{(x-Z_k)(\bar{w}-\bar{Z}_k)}{|Y(Z_k)|^2}\,\frac{1}{T^5\prod_{i<j=1}^{6}|A_{ij}|^2}- \\ &\qquad\quad-\frac{1}{4}\sum_{i=1}^{6}\frac{\partial}{\partial A_i}\left[\left(\frac{1}{A_i-S_A(r)}-\frac{1}{A_i-x}\right)\times\right. \\ &\qquad\quad\left.\times\prod_{l=1}^{4}\frac{(A_i-Z_l)(\bar{S}_A(\bar{s})-\bar{Z}_l)}{|Y(Z_l)|^2}\,\frac{\bar{I}(\bar{S}_A(\bar{s}))}{T^5\prod_{j<k=1}^{6}|A_{jk}|^2}\right]- \\ &\qquad\quad-\frac{1}{4}\sum_{i=1}^{6}\frac{\partial}{\partial\bar{A}_i}\left[\left(\frac{1}{\bar{A}_i-\bar{S}_A(\bar{s})}-\frac{1}{\bar{A}_i-\bar{w}}\right)\times\right. \\ &\qquad\quad\left.\times\prod_{l=1}^{4}\frac{(\bar{A}_i-\bar{Z}_l)(x-Z_l)}{|Y(Z_l)|^2}\,\frac{I_M(x)}{T^5\prod_{j<k=1}^{6}|A_{jk}|^2}\right].\end{aligned} \tag{B.11}$$

In deriving the above equation we have used the fact that the only non-holomorphic (meromorphic) part in both $I(x)$ and $I_M(x)$ is from the terms $-\frac{5}{4}\sum_{i=1}^{6}\frac{1}{x-A_i}\frac{\partial}{\partial A_i}\ln T$, i.e. T depends on both A_i and $\bar{A}_i$.

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